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THE BUCKLING OF LATTICE COLUMNS WITH STOCHASTIC IMPERFECTIONS

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Abstract—An analysis is presented for determining the buckling load of triangular lattice columns with combined local and overall imperfections. For the case where the imperfections are deterministic and uniform, the nonlinear problem is solved in terms of quadratures. The resulting buckling loads are shown to compare favorably with the predictions of a straightforward single-term Ritz approximation. The Ritz approach is used to derive estimators for the mean and standard deviation of the buckling load for the situation where the local imperfections are stochastic. The resulting estimators are shown to be valid by comparing their results with those obtained by a Monte Carlo simulation.

1. INTRODUCTION

Efficiently designed compression columns for carrying small loads tend to be slender and to be built up of lattice members which are themselves slender. Such columns are known to be sensitive to initial imperfections. A slight curvature in the overall centerline, for example, causes the compressive end load to induce increased loads in the local members. The result is premature crippling of those local members. On the other hand, local imperfections of the cross section produce a reduction of the effective column bending stiffness and thereby induce premature column buckling. Clearly the proper design of such slender lattice columns must take into account the simultaneous effects of local and overall imperfections.

The problem of imperfection sensitivity of slender columns has received a great deal of attention in the past. However, most studies have dealt only with imperfections in the overall centerline of the column. Only relatively recently has the problem of local imperfections in columns been treated. In a study by van der Neut[1], the effects were determined of combined local and overall deterministic imperfections on the buckling load of a column built up from thin plates. Crawford and Hedgepath[2] investigated the effects of deterministic local imperfections on the buckling of lattice columns and sandwich panels.

The important problem of combined local and overall imperfections in lattice columns has been neglected. Furthermore, those imperfections that arise from material or fabrication variations (as opposed to deterministic environmental effects such as lateral loading and thermal gradients) tend to be random in nature. While such random components in the overall imperfections can be made much less important than the deterministic components by careful fabrication, the local imperfections often remain predominantly random in spite of such efforts. Therefore, proper design requires the treatment of combined effects in a statistical manner.

Among the many statistical analyses of imperfection sensitive columns, the majority are concerned with the buckling of a simple uniform column with an initial centerline deflection which is random in some way. In an early study of this problem, Boyce[3] considered a simple uniform column with an initial deflection in the form of a stationary random function of position along the column. Fraser and Budiansky[4], Amazigo, Budiansky and Carrier[5], Amazigo[6] and Videc and Sanders[7] have applied various methods to obtain asymptotic expressions for the buckling load of a uniform column with stationary random initial deflections and resting on a nonlinear elastic foundation. Bernard and Bogdanoff[8] considered a uniform column with random initial deflections in two directions and a random initial twist. Jacquot[9]

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applied a Green's function technique to a uniform column with nonstationary random initial deflections. Roorda[10] considered an ensemble of uniform columns with initial deflection in the form of a half sine wave, and with eccentrically applied load. The amplitudes of initial deflections and load eccentricities were considered to be jointly random variables. None of the studies just cited considers the effects of local imperfections.

The primary objective of this paper is the analysis of the combined effects of random local imperfections and deterministic overall imperfections on the buckling load of triangular lattice columns. In a preliminary deterministic analysis, the combined effects of local and overall imperfections on the buckling load are calculated for a broad range of imperfection amplitudes. In the more general case of random local imperfections, estimators for the mean and standard deviation of the random buckling load are developed. These estimators are shown to provide an accurate basis for predicting the minimum $3-\sigma$ (99.9%) buckling load.

2. FORMULATION

Consider the simply supported triangular lattice column of length L shown in Fig. 1. The column consists of three uniform longerons located on a circle of radius R about the column's center. Each longeron has cross-sectional area A and minimum bending modulus EI. The column is divided into bays of length $l \ll L$ by an assemblage of battens and diagonals that stabilize the cross section and provide shear stiffness.

For the purposes of the present study, the effects of cross-sectional and transverse-shear deformations are ignored. Then the geometry of the deformed column is defined by the location of its centerline which is coincident with the center of each triangular batten assembly. Furthermore, the familiar assumptions of elementary engineering beam theory apply.

If a compressive load P is applied to a perfect column, buckling will occur when P equals the classical buckling load P_c, which is the smaller of the local buckling load P_l and the overall buckling load P_E where

$$P_l = \frac{3\pi^2 EI}{l^2} \tag{1}$$

$$P_E = \frac{3\pi^2 EAR^2}{2L^2} \tag{2}$$

For loads smaller than P_c, the column remains straight, both locally and overall. Bifurcation occurs at the buckling load, P_c.

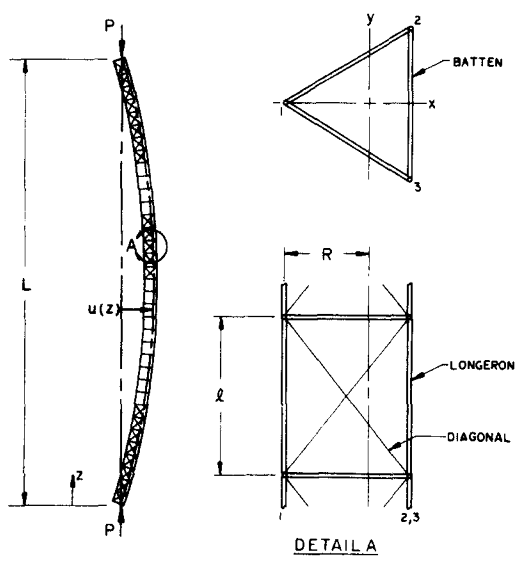

Fig. 1. Triangular lattice column and coordinates.

For an imperfect column, on the other hand, the column deflects laterally as the compressive load is increased. If both local and overall imperfections are present, the process continues until the compressive load reaches a maximum value; further compression of the column produces further lateral deflection but a decrease in the axial compressive load. In this paper, the maximum load so obtained is defined as the "buckling" load even though buckling, in the bifurcation sense, never occurs. The objective of this analysis is to calculate that buckling load.

Referring to Fig. 1, let z be the axis which joins the column ends, and let x and y be axes which are orthogonal to z and aligned as shown. Let the longerons be numbered 1, 2, 3 as shown.

Let $u(z)$ and $v(z)$ be the centerline deflection in the x and y directions, respectively. Let $u_0(z)$ and $v_0(z)$ locate the initial deformed centerline before the application of the compressive load. Then the curvature relations may be expressed as

$$\left.\begin{aligned} \frac{d^2u(z)}{dz^2} &= -\frac{2\epsilon_1(z)-[\epsilon_2(z)+\epsilon_3(z)]}{3R}+\frac{d^2u_0(z)}{dz^2} \\ \frac{d^2v(z)}{dz^2} &= \frac{\epsilon_2(z)-\epsilon_3(z)}{\sqrt{(3)}R}+\frac{d^2v_0(z)}{dz^2} \end{aligned}\right\} \tag{3}$$

where $\epsilon_i(z)$ is the unit shortening of the ith longeron ($i = 1, 2, 3$).

In developing these curvature relations it has been assumed that the bay length, l, is infinitesimal in comparison with the overall column length, L, so that the unit shortening $\epsilon_i(z)$ may be treated as a continuous variable. Of course $\epsilon_i(z)$ is actually a discrete variable, being constant within each bay of the column. The value of $\epsilon_i(z)$ in this formulation is defined as the unit shortening of the ith longeron in a generic bay whose center is located at position z along the column axis. In the limit as $l/L \to 0$ the curvature relations become exact.

Let $p_i(z)$ be the compressive load in the ith longeron in the bay located at position z. Then, requiring the column to remain in static equilibrium it may be shown that

$$\left.\begin{aligned} p_1(z) &= (P/3)[1+2u(z)/R] \\ p_{2,3}(z) &= (P/3)[1-u(z)/R \mp v(z)\sqrt{(3)}/R]. \end{aligned}\right\} \tag{4}$$

Let the local imperfection of the ith longeron in the bay located at z be in the form of a half sine wave with amplitude $a_i(z)$. Furthermore, assume that each longeron deflects as if it were a simply supported column of length l. Then the unit shortening can be obtained as

$$\epsilon_i(z) = \frac{P_l}{3EA}\left(\frac{3p_i(z)}{P_l}+\frac{a_i^2(z)A}{4I}\left\{\left[1-\frac{3p_i(z)}{P_l}\right]^{-2}-1\right\}\right); \quad i = 1, 2, 3 \tag{5}$$

where the second term represents the unit shortening due to the lateral deflection induced by the axial load.

Equations (3)–(5) provide a set of second-order nonlinear differential equations for the centerline deflections $u(z)$ and $v(z)$ as functions of the compressive load P, overall imperfections $u_0(z)$ and $v_0(z)$, and local imperfections $a_i(z)$. The appropriate boundary conditions are

$$u(0) = v(0) = u(L) = v(L) = 0. \tag{6}$$

Nondimensionalization

In order to reduce the number of parameters appearing in the governing equations and to facilitate numerical solution, the following nondimensional quantities are introduced:

$$\begin{aligned} &\zeta = z/L; \qquad \xi = u/R; \qquad \eta = v/R; \\ &\bar{\epsilon}_i = 3EA\epsilon_i/P_l; \qquad \bar{p}_i = 3p_i/P_l; \\ &\bar{P} = P/P_l; \qquad \delta_i = (A/I)^{1/2}a_i. \end{aligned} \tag{7}$$

Equations (3)–(5) become

$$\left.\begin{aligned}\frac{d^2\xi}{d\zeta^2}+\pi^2\frac{P_l}{P_E}\frac{(2\bar{\epsilon}_1-\bar{\epsilon}_2-\bar{\epsilon}_3)}{6}=\frac{d^2\xi_0}{d\zeta^2}\\ \frac{d^2\eta}{d\zeta^2}+\pi^2\frac{P_l}{P_E}\frac{(\bar{\epsilon}_3-\bar{\epsilon}_2)}{2\sqrt{3}}=\frac{d^2\eta_0}{d\zeta^2}\end{aligned}\right\} \tag{8}$$

$$\bar{\epsilon}_i=\bar{p}_i+\delta_i^2[(1-\bar{p}_i)^{-2}-1]/4; \qquad i=1,2,3 \tag{9}$$

$$\bar{p}_1=\bar{P}(1+2\xi); \qquad \bar{p}_{2,3}=\bar{P}(1-\xi\mp\eta\sqrt{3}) \tag{10}$$

and the boundary conditions are

$$\xi(0)=\eta(0)=\xi(1)=\eta(1)=0. \tag{11}$$

3. ANALYSIS FOR UNIFORM DETERMINISTIC IMPERFECTIONS

Considered in this section are the combined effects of local and overall deterministic imperfections on the buckling load of the lattice column. It is assumed that the overall centerline of the column has an initial imperfection in the shape of a parabola. It is further assumed that this parabola is aligned along the worst-case direction in which longeron "1" is most highly compressed as shown in Fig. 1. Thus $\eta_0=0$. Let the initial center deflection be

$$u_0(L/2)=R\Delta/\sqrt{2}. \tag{12}$$

Then

$$\frac{d^2\xi_0}{d\zeta^2}=-4\sqrt{(2)}\Delta$$

is a constant for all ζ. Note that the imperfection amplitude Δ is expressed here in terms of the column radius of gyration in order to preserve similarity with the treatment of the local imperfections.

The local imperfection amplitudes δ_i are assumed to be uniform over the length of the column and the same for all longerons. Therefore

$$\delta_i(\zeta)=\delta \qquad (i=1,2,3) \tag{13}$$

where δ is constant.

With these assumptions, the column deflects only in the x-direction and the boundary-value problem for the nondimensional centerline deflection becomes

$$\frac{d^2\xi}{d\zeta^2}+\pi^2\frac{P_l}{P_E}\left\{\bar{P}\xi+\frac{\delta^2}{12}[1-\bar{P}(1+2\xi)]^{-2}-\frac{\delta^2}{12}[1-\bar{P}(1-\xi)]^{-2}\right\}=-4\sqrt{(2)}\Delta$$
$$\xi(0)=\xi(1)=0. \tag{14}$$

For the special case of a perfect column with $\delta=\Delta=0$, this boundary-value problem reduces to the familiar problem which governs Euler buckling. In that case, of course the solution $\xi(\zeta)$ would be a half-sine wave over the column length. For an imperfect column with nonzero δ and Δ, the solution is expected to be qualitatively similar; the centerline deflection has its maximum at $\zeta=1/2$ and decreases monotonically and symmetrically to zero at the ends.

Multiplying the first of eqns (14) by $d\xi/d\zeta$, integrating, solving the results for $d\zeta/d\xi$, and integrating with respect to ξ from $\xi=0$ to the midspan value $\bar{\xi}$ yields an integral expression for the half-length of the column, which may be expressed as

$$\frac{1}{2}=\frac{1}{\pi}(P_E/P)^{1/2}\int_0^{\bar{\xi}}[f(\xi)-f(\bar{\xi})]^{-1/2}\,d\xi \tag{15}$$

where

$$f(\xi) = -\frac{8\sqrt{2}}{\pi^2}\frac{P_E}{P_l}\Delta\xi - \bar{P}\xi^2 - \frac{\delta^2}{12\bar{P}}\{[1-\bar{P}(1+2\xi)]^{-1} + 2[1-\bar{P}(1-\xi)]^{-1}\}. \tag{16}$$

For a given maximum deflection $\bar{\xi}$ of the column centerline, imperfection amplitudes δ and Δ, and geometric parameter P_E/P_l, eqn (15) may be solved by numerical techniques for the corresponding compressive load $\bar{P}$. The load-deflection curve for the column may then be generated by solving for the values of the load $\bar{P}$ which correspond to successively increasing mid-span deflections $\bar{\xi}$. The buckling load $\bar{P}_b$ for the imperfect column is defined as that load which corresponds to the peak in the load-deflection curve. As an illustration, several load-deflection curves obtained by such an exact simulation of the column behavior are shown in Fig. 2. Each curve corresponds to a lattice column with different geometric properties but with the same nondimensional imperfection amplitudes δ and Δ. The buckling loads $\bar{P}_b$ are identified in the figure with enlarged circles.

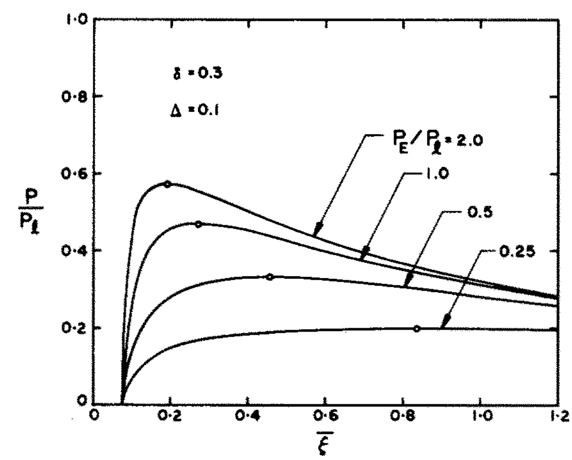

Fig. 2. Example load-deflection curves. The circle symbols identify the buckling loads.

Once the compressive load $\bar{P}$ has been determined for a given mid-span deflection $\bar{\xi}$, the corresponding deflected shape $\xi(\zeta)$ may be directly determined. Before the column is loaded, the deflected shape is that of the assumed initial parabola. As the load is progressively increased, $\xi(\zeta)$ becomes more sharply curved at midspan, approaching a triangular shape (with local crimping at midspan) for very large deflections. It was found in simulations of several test columns that the deflected shape is very closely approximated by a half sine wave as $\bar{P}$ approaches the buckling load $\bar{P}_b$. This noteworthy observation was found to be valid for all imperfection amplitudes and column geometries investigated.

The combined effects of local and overall imperfections on the buckling load are shown in Figs. 3(a)–(c) for a wide range of column geometries and imperfection amplitudes. The results are plotted as a function of the geometric parameter P_E/P_l. The ordinate is the ratio of the buckling load of the imperfect column to that of the perfect column, P_c, which is the smaller of P_E and P_l. The curves, therefore, directly express the reduction in strength due to the imperfections. The curves in each figure correspond to the same local imperfection amplitude δ and different overall imperfection amplitudes Δ.

Several trends may be observed in Figs. 3(a)–(c). Clearly, the buckling load is reduced as either δ or Δ is increased. The buckling load is most sensitive to imperfections when $P_E = P_l$.

In order to illustrate the relative sensitivity to local and overall imperfections, the contour lines of constant strength reduction factor are plotted vs $\delta^{1/2}$ and $\Delta^{1/2}$ in Fig. 4 for the most sensitive case of $P_E = P_l$. The square roots are used because of the square-root dependence of the buckling load for small imperfections. The buckling load is slightly more sensitive to Δ than to δ.

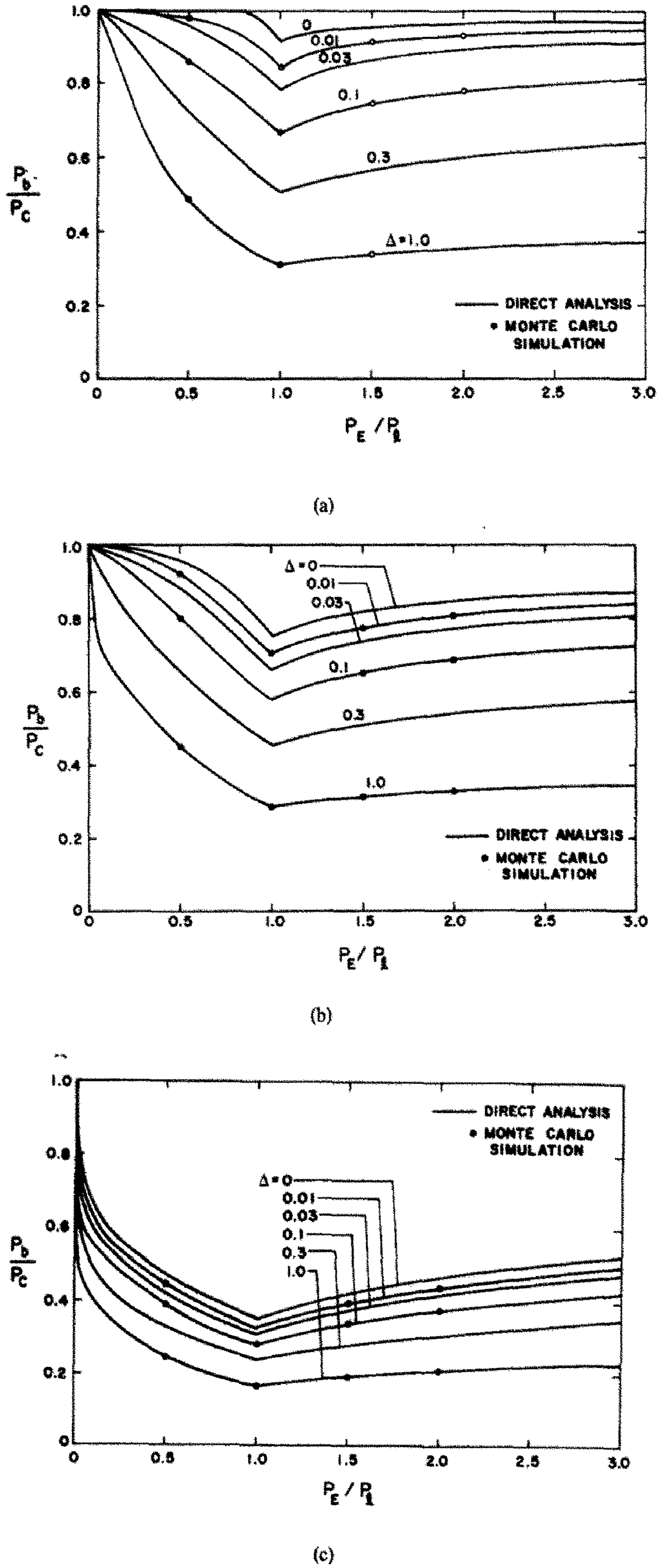

Fig. 3. Variation of buckling load with ratio of Euler (overall) to local buckling load. (a) δ or σ_δ equals 0.01. (b) δ or σ_δ equals 0.1. (c) δ or σ_δ equals 1.0.

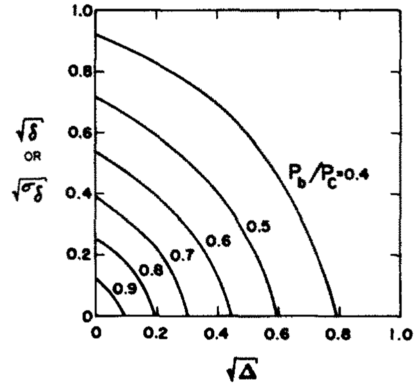

Fig. 4. Interaction diagram for the combined effect of local and overall imperfections on the buckling load.

4. RITZ APPROXIMATION

In order to facilitate the analysis of random imperfections, a simplified approximate technique for solving the boundary-value problem is desirable. The close similarity between the exact deflection shape at buckling and a half sine wave observed in the preceding section leads to the expectation that a single-term Ritz procedure would yield accurate values of the buckling load.

The variational equivalent to the boundary-value problem of eqns (8)–(10) is

$$\delta\Phi = 0 \tag{17}$$

where

$$\Phi = \frac{1}{2}\int_0^1 \Bigg(-\frac{P_E}{\pi^2 P_l}\left\{\left[\frac{d(\xi-\xi_0)}{d\zeta}\right]^2 + \left[\frac{d(\eta-\eta_0)}{d\zeta}\right]^2\right\} + P(\xi^2+\eta^2) - \frac{2\delta_1{}^2-\delta_2{}^2-\delta_3{}^2}{12}\xi - \frac{\delta_3{}^2-\delta_2{}^2}{4\sqrt{3}}\eta + \frac{\delta_1{}^2}{12\bar{P}}[1-\bar{P}(1+2\xi)]^{-1} + \frac{\delta_2{}^2}{12\bar{P}}[1-\bar{P}(1-\xi-\eta\sqrt{3})]^{-1} + \frac{\delta_3{}^2}{12\bar{P}}[1-\bar{P}(1-\xi+\eta\sqrt{3})]^{-1}\Bigg)\,d\zeta. \tag{18}$$

Let the deflection shapes be given by the aforementioned approximation

$$\begin{aligned}\xi &= \bar{\xi}\sin\pi\zeta \\ \eta &= \bar{\eta}\sin\pi\zeta\end{aligned} \tag{19}$$

where $\bar{\xi}$ and $\bar{\eta}$ represent the undetermined nondimensional midspan deflections in the x and y directions respectively. Also, assume once again that the overall imperfection is a deterministic parabolic deflection in the worst case direction with a midspan amplitude given by eqn (12). Finally, wherever closed-form integration is not possible, the integral may be approximated by dividing the length into N sub-intervals, evaluating the integrand at the midpoint of each sub-interval, and summing. The result is

$$\Phi = -\frac{1}{4}\left(\frac{P_E}{P_l}-\bar{P}\right)(\bar{\xi}^2+\bar{\eta}^2) + \frac{8\sqrt{2}}{\pi^3}\frac{P_E}{P_l}\Delta\bar{\xi} - \frac{4}{3\pi^2}\frac{P_E}{P_l}\Delta^2 + \frac{1}{24N}\sum_{j=1}^{N}\Big\{-(2\delta_{1j}^2-\delta_{2j}^2-\delta_{3j}^2)s_j\bar{\xi} - \sqrt{(3)}(\delta_{3j}^2-\delta_{2j}^2)s_j\bar{\eta} + \frac{\delta_{1j}^2}{\bar{P}}[1-\bar{P}(1+2\bar{\xi}s_j)]^{-1} + \frac{\delta_{2j}^2}{\bar{P}}[1-\bar{P}(1-\bar{\xi}s_j-\bar{\eta}s_j\sqrt{3})]^{-1} + \frac{\delta_{3j}^2}{\bar{P}}[1-\bar{P}(1-\bar{\xi}s_j+\bar{\eta}s_j\sqrt{3})]^{-1}\Big\} \tag{20}$$

where

$$s_j \equiv \sin\frac{j-1/2}{N}\pi$$
$$\delta_{ij} \equiv \delta_i\left(\frac{j-1/2}{N}\right). \tag{21}$$

Minimizing Φ with respect to the undetermined parameters is accomplished by setting the derivatives of Φ with respect to $\bar{\xi}$ and $\bar{\eta}$ equal to zero. The resulting equations may be expressed as

$$\left(\frac{P_E}{P_l}-\bar{P}\right)\bar{\xi}=\frac{16\sqrt{2}}{\pi^3}\frac{P_E}{P_l}\Delta-\frac{1}{12N}\sum_{j=1}^{N}(2\delta_{1j}^2-\delta_{2j}^2-\delta_{3j}^2)s_j+\frac{1}{12N}\sum_{j=1}^{N}s_j\{2\delta_{1j}^2[1-\bar{P}(1+2\bar{\xi}s_j)]^{-2}$$
$$-\delta_{2j}^2[1-\bar{P}(1-\bar{\xi}s_j-\bar{\eta}s_j\sqrt{3})]^{-2}-\delta_{3j}^2[1-\bar{P}(1-\bar{\xi}s_j+\bar{\eta}s_j\sqrt{3})]^{-2}\}, \tag{22}$$

$$\left(\frac{P_E}{P_l}-\bar{P}\right)\bar{\eta}=\frac{\sqrt{3}}{12N}\sum_{j=1}^{N}(\delta_{2j}^2-\delta_{3j}^2)s_j+\frac{\sqrt{3}}{12N}\sum_{j=1}^{N}s_j\{-\delta_{2j}^2[1-\bar{P}(1-\bar{\xi}s_j-\bar{\eta}s_j\sqrt{3})]^{-2}$$
$$+\delta_{3j}^2[1-\bar{P}(1-\bar{\xi}s_j+\bar{\eta}s_j\sqrt{3})]^{-2}\}. \tag{23}$$

For given P_E/P_l, Δ, and δ_{ij}, eqns (22) and (23) can be solved by numerical techniques for the one-dimensional continuum of values of $\bar{P}$, $\bar{\xi}$ and $\bar{\eta}$. Thus, in the manner described in the previous section the entire load-deflection curve, and in particular the buckling load $\bar{P}_b$, and corresponding midspan deflections $\bar{\xi}_b$ and $\bar{\eta}_b$, may be obtained for a lattice column with arbitrarily specified local imperfection amplitudes δ_{ij}.

In the special case when each δ_{ij} equals the same constant δ, the situation reduces to that of uniform deterministic local imperfections. Equation (23) is then satisfied by setting $\bar{\eta}=0$ and eqn (22) becomes

$$\left(\frac{P_E}{P_l}-\bar{P}\right)\bar{\xi}=\frac{16\sqrt{2}}{\pi^3}\frac{P_E}{P_l}\Delta+\frac{\delta^2}{6N}\sum_{j=1}^{N}s_j\{[1-\bar{P}(1+2\bar{\xi}s_j)]^{-2}-[1-\bar{P}(1-\bar{\xi}s_j)]^{-2}\}. \tag{24}$$

As a check on the accuracy of the Ritz approximate solution in this special case, the buckling loads for several test columns with widely varying parameters were determined from eqn (24) and compared with those obtained by the exact solution determined from eqn (16). For each test column the error in the Ritz approximate solution was less than 0.5%. The approach can therefore be used with confidence for the more general problem of random imperfections.

5. ANALYSIS FOR RANDOM LOCAL IMPERFECTIONS

Attention is now turned to the case in which the local imperfection amplitudes δ_{ij} are random variables. Although these imperfections are nonuniform over the length of any particular column, the statistical properties can be expected to be invariant inasmuch as the material and fabrication techniques can be expected to be invariant for the three longerons and various sections of the column. A situation of practical interest is the case in which the δ_{ij} are considered to be independent samples of the same Gaussian random process with zero mean and standard deviation σ_δ. It then follows that

$$\langle\delta_{ij}^2\rangle=\sigma_\delta^2 \tag{25}$$

where the brackets $\langle\cdot\rangle$ represent the "expectation" or ensemble average.

Since the δ_{ij} are random variables, it follows that $\bar{P}$, $\bar{\xi}_b$ and $\bar{\eta}_b$ are also random variables. Thus, the objective of the following analysis is the determination of the statistical properties of the resulting random buckling load $\bar{P}_b$. One method of doing this would be to consider an ensemble of test columns, determine the buckling load of each one, and then construct the probability distribution for the buckling load. Such a Monte Carlo analysis was indeed performed, as is discussed in the next section. However, a more direct and less arduous method of estimating the statistical properties is clearly desirable.

In order to approach the estimation of buckling-load statistics, consider $\bar{\xi}$ to be a specified deterministic quantity. Then eqns (22) and (23) may be used to determine the random variables $\bar{P}$ and $\bar{\eta}$ in terms of $\bar{\xi}$. In particular, the ensemble average and the standard deviation of $\bar{P}$ for fixed $\bar{\xi}$ may be determined, at least in principle.

Select $\bar{\xi}$ to be that value for which the ensemble average of $\bar{P}$ is a maximum. Because the load deflection curves (see Fig. 2) have broad maxima, the statistics of the load $\bar{P}$ corresponding to this fixed value of $\bar{\xi}$ can be expected to be nearly the same as the statistics of the buckling load $\bar{P}_b$ obtained by properly treating $\bar{\xi}$ as a random variable. The assumption is therefore made that the desired results can be obtained by this fixed $-\bar{\xi}$ approach.

One characteristic of the Monte Carlo results reported later is that the loads for fixed $\bar{\xi}$ fluctuate only slightly. This characteristic suggests that useful results can be obtained by linearizing the governing equations (22) and (23) about the average values. Let

$$\begin{aligned} \bar{P} &= \langle \bar{P} \rangle + q \\ \delta_{ij}^2 &= \sigma_\delta^2 + \beta_{ij} \end{aligned} \tag{26}$$

and consider q, β_{ij} and $\bar{\eta}$ to be small quantities. To zeroth order in these quantities eqn (23) is identically zero and eqn (22) becomes exactly the same as eqn (24) with $\bar{P}$ replaced by $\langle \bar{P} \rangle$ and δ by σ_δ.

To first order, eqns (22) and (23) give

$$\begin{aligned} &\left(\bar{\xi} + \frac{\sigma_\delta^2}{3N} \sum_{j=1}^{N} s_j \{(1+2\bar{\xi}s_j)[1-\langle \bar{P} \rangle(1+2\bar{\xi}s_j)]^{-3} - (1-\bar{\xi}s_j)[1-\langle \bar{P} \rangle(1-\bar{\xi}s_j)]^{-3}\}\right) q \\ &= \frac{1}{12N} \sum_{j=1}^{N} s_j(2\beta_{1j} - \beta_{2j} - \beta_{3j}) - \frac{1}{12N} \sum_{j=1}^{N} s_j \{2\beta_{1j}[1-\langle \bar{P} \rangle(1+2\bar{\xi}s_j)]^{-2} \\ &\quad - (\beta_{2j}+\beta_{3j})[1-\langle \bar{P} \rangle(1-\bar{\xi}s_j)]^{-2}\}, \end{aligned} \tag{27}$$

$$\left(\frac{P_E}{P_l} - \langle \bar{P} \rangle - \frac{\sigma_\delta^2 \langle \bar{P} \rangle}{N} \sum_{j=1}^{N} s_j^2 [1-\langle \bar{P} \rangle(1-\bar{\xi}s_j)]^{-3}\right) \bar{\eta} = \frac{\sqrt{3}}{12N} \sum_{j=1}^{N} s_j(\beta_{2j} - \beta_{3j})\{1-[1-\langle \bar{P} \rangle(1-\bar{\xi}s_j)]^{-2}\}. \tag{28}$$

Taking expectations of these two equations (note that the coefficients on the left-hand side are deterministic) yields the result that

$$q = \bar{\eta} = 0$$

which is consistent with the linearization about the averages.

The foregoing discussion indicates that an estimator for the mean value of the buckling load may be determined by analysis of the much simpler case of uniform deterministic imperfections by choosing the deterministic imperfection amplitude δ to equal the standard deviation of the random imperfection amplitudes σ_δ. Therefore Figs. 3 and 4 can be used directly to estimate the average buckling load for random imperfections.

In order to determine the degree of scatter of the buckling loads, an estimator is required for the standard deviation of the buckling load $\sigma_{\bar{P}_b}$. Such an estimator can be obtained directly from eqn (27) by squaring and averaging, while selecting $\bar{\xi} = \bar{\xi}_b$ and $\langle \bar{P} \rangle = \bar{P}_b$, where $\bar{P}_b$, $\bar{\xi}_b$ are determined from eqn (24) with $\delta = \sigma_\delta$.

The averaging process requires the determination of $\langle \beta_{ij} \beta_{mn} \rangle$. In this analysis the quantities δ_{ij} and δ_{mn} are taken to be statistically independent for $j \neq n$. This can be accomplished by selecting the numerical-integration interval L/N at least as large as the effective correlation interval of the imperfection. In columns for which the longerons are pinned at each batten set, the bay length l can be used. For columns with continuous longerons the correlation interval may be several bays in length.

Within the correlation interval, the value of $\langle\beta_{ij}\beta_{nj}\rangle$ depends on whether δ_{ij} and δ_{nj} are considered to be independent of each other. If within the correlation interval each longeron is assumed to have the same local imperfection, then

$$\langle\beta_{ij}\beta_{nj}\rangle = 2\sigma_\delta^4$$

and (29)

$$\sigma_{\bar{P}_b} = \sigma_\delta^2 C_1^{1/2}/|\bar{\xi}_b + \sigma_\delta^2 C_0|$$

where

$$C_0 = \frac{1}{3N}\sum_{j=1}^{N} s_j\{(1+2\bar{\xi}_b s_j)[1-\bar{P}_b(1+2\bar{\xi}_b s_j)]^{-3} - (1-\bar{\xi}_b s_j)[1-\bar{P}_b(1-\bar{\xi}_b s_j)]^{-3}\}, \tag{30}$$

$$C_1 = \frac{1}{18N^2}\sum_{j=1}^{N} s_j^2\{[1-\bar{P}_b(1+2\bar{\xi}_b s_j)]^{-2} - [1-\bar{P}_b(1-\bar{\xi}_b s_j)]^{-2}\}^2. \tag{31}$$

If, on the other hand, complete independence of the local imperfections is assumed

$$\sigma_{\bar{P}_b} = \sigma_\delta^2 C_2^{1/2}/|\bar{\xi}_b + \sigma_\delta^2 C_0| \tag{32}$$

where

$$C_2 = \frac{1}{36N^2}\sum_{j=1}^{N} s_j^2\left(2\{[1-\bar{P}_b(1+2\bar{\xi}_b s_j)]^{-2} - 1\}^2 + \{[1-\bar{P}_b(1-\bar{\xi}_b s_j)]^{-2} - 1\}^2\right). \tag{33}$$

Equations (29) or (32) give the desired estimator for the standard deviation of the buckling load. Values have been calculated for a large range of parameters and are shown in Table 1 for both types of independence. Note that the values of $\sigma_{\bar{P}_b}$ are at least an order of magnitude smaller than $\bar{P}_b$ and usually over two orders of magnitude smaller.

6. MONTE CARLO SIMULATION

In order to guide the analysis and to assess the validity of the estimators derived in the preceding section, a Monte Carlo simulation study was performed. Considered in the study were a broad range of material and geometric parameters P_E and P_l, and deterministic overall imperfection Δ. For each set of values of these parameters an ensemble of 200 lattice columns were selected by choosing the set of local imperfection amplitudes $\delta_{1j} = \delta_{2j} = \delta_{3j}$ ($j = 1, 2, 3, \ldots N$) for each column to be different independent samples of the same Gaussian random process with zero mean and standard deviation σ_δ. In every case N was chosen as 225. An ensemble of Ritz approximate solutions for the buckling loads were then determined from eqn (22) for a wide range of random local imperfection amplitudes. The results for the mean values of the buckling loads are shown as enlarged circles in Figs. 3(a)–(c). The agreement with the results for uniform deterministic imperfections is very good.

The means and standard deviations obtained from the Monte Carlo simulations are given in Table 2. One meaningful way to compare these results with the predictions of the estimators (Table 1) is by determining the minimum $(3-\sigma)$ buckling load, i.e. the load which can be exceeded without buckling in approx. 99.9% of the sample columns within an ensemble, and which is often sought in statistical structural design. On this basis the estimator technique is accurate to within 1.5% for all cases. The direct estimators are therefore valid and useful.

7. CONCLUSIONS

An analysis is presented for determining the combined effects of local and overall imperfections on the buckling load of triangular lattice columns. The imperfections considered are due to an initial curvature of the local compression members and the overall column axis. In a

Table 1. Mean and standard deviation of the buckling load as predicted by the statistical estimators

σ_δ	Δ	P_E/P_ℓ	$\langle\bar{P}_b\rangle$ Eqn (24)	$\sigma_{\bar{P}_b}$ Eqn (29)	$\sigma_{\bar{P}_b}$ Eqn (32)
0.01	0.01	0.5	0.490	0.000174	0.00176
0.01	0.01	1.0	0.845	0.00193	0.00207
0.01	0.01	1.5	0.916	0.00187	0.00211
0.01	0.01	2.0	0.932	0.00161	0.00182
0.01	0.10	0.5	0.436	0.000390	0.000390
0.01	0.10	1.0	0.666	0.00124	0.00125
0.01	0.10	1.5	0.748	0.00150	0.00151
0.01	0.10	2.0	0.782	0.00145	0.00146
0.01	1.00	0.5	0.248	0.000193	0.000193
0.01	1.00	1.0	0.313	0.000349	0.000348
0.01	1.00	1.5	0.340	0.000426	0.000426
0.10	0.01	0.5	0.461	0.00222	0.00281
0.10	0.01	1.0	0.704	0.00616	0.00904
0.10	0.01	1.5	0.777	0.00611	0.0103
0.10	0.01	2.0	0.809	0.00559	0.00928
0.10	0.10	0.5	0.399	0.00243	0.00252
0.10	0.10	1.5	0.651	0.00509	0.00546
0.10	0.10	2.0	0.687	0.00504	0.00543
0.10	1.00	0.5	0.226	0.00154	0.00153
0.10	1.00	1.0	0.285	0.00195	0.00195
0.10	1.00	1.5	0.314	0.00198	0.00198
0.10	1.00	2.0	0.328	0.00202	0.00202
1.00	0.01	0.5	0.233	0.00912	0.0207
1.00	0.01	1.5	0.396	0.0129	0.0330
1.00	0.01	2.0	0.440	0.0131	0.0342
1.00	0.10	0.5	0.195	0.00782	0.00942
1.00	0.10	1.0	0.285	0.0102	0.0131
1.00	0.10	1.5	0.340	0.0111	0.0145
1.00	0.10	2.0	0.377	0.0114	0.0149
1.00	1.00	0.5	0.123	0.00441	0.00425
1.00	1.00	1.0	0.166	0.00525	0.00517
1.00	1.00	1.5	0.191	0.00554	0.00549
1.00	1.00	2.0	0.208	0.00563	0.00561

Table 2. Mean and standard deviation of the buckling load as predicted by Monte Carlo simulation

σ_δ	Δ	P_E/P_ℓ	$\langle\bar{P}_b\rangle$	$\sigma_{\bar{P}_b}$
0.01	0.01	0.5	0.490	0.000177
0.01	0.01	1.0	0.846	0.00185
0.01	0.01	1.5	0.916	0.00181
0.01	0.01	2.0	0.933	0.00156
0.01	0.10	0.5	0.436	0.000432
0.01	0.10	1.0	0.665	0.00123
0.01	0.10	1.5	0.747	0.00143
0.01	0.10	2.0	0.782	0.00141
0.01	1.00	0.5	0.249	0.000386
0.01	1.00	1.0	0.313	0.000511
0.01	1.00	1.5	0.340	0.000532
0.10	0.01	0.5	0.462	0.00207
0.10	0.01	1.0	0.705	0.00588
0.10	0.01	1.5	0.778	0.00575
0.10	0.01	2.0	0.810	0.00529
0.10	0.10	0.5	0.399	0.00232
0.10	0.10	1.5	0.652	0.00487
0.10	0.10	2.0	0.688	0.00478
0.10	1.00	0.5	0.227	0.00151
0.10	1.00	1.0	0.287	0.00189
0.10	1.00	1.5	0.314	0.00197
0.10	1.00	2.0	0.329	0.00196
1.00	0.01	0.5	0.224	0.00854
1.00	0.01	1.5	0.398	0.0123
1.00	0.01	2.0	0.442	0.0125
1.00	0.10	0.5	0.196	0.00734
1.00	0.10	1.0	0.287	0.00975
1.00	0.10	1.5	0.341	0.0105
1.00	0.10	2.0	0.379	0.0108
1.00	1.00	0.5	0.123	0.00415
1.00	1.00	1.0	0.167	0.00498
1.00	1.00	1.5	0.192	0.00522
1.00	1.00	2.0	0.209	0.00530

preliminary deterministic analysis, the case of uniform local imperfections is investigated. In particular, results are presented for the dependence of the deterministic buckling load on the various material, geometric and imperfection parameters.

An approximate analysis for the more general case of random local imperfections is developed on the basis of a single-term Ritz approach. Based on this analysis, an estimator for the mean value of the random buckling load is presented. This estimator turns out to be identical with the buckling load of a comparison column with uniform deterministic local imperfections, provided that the amplitude of local imperfections in the comparison column is set equal to the standard deviation of random local imperfections.

In order to evaluate the statistical scatter of the buckling load, two estimators for its standard deviation are developed from a linearization about the mean value. The predictions of the estimators for the mean and standard deviation of the random buckling load are presented for various material, geometric and imperfection parameters.

Also presented are the results of a Monte Carlo simulation study used to assess the validity of the simplified statistical estimators. From a comparison of the results of the Monte Carlo simulation and the Ritz approximate analysis, it is concluded that the estimators provide an accurate basis for predicting the minimum $(3-\sigma)$ buckling load for use in probabilistic structural design.

Although consideration has been confined in this paper to triangular lattice columns, the basic approach could be successfully applied to a variety of imperfection-sensitive structures.

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